Low energy collective modes, Ginzburg-Landau theory, and pseudogap behavior in superconductors with long-range pairing interactions

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We study the superconducting instability in systems with long but finite ranged, attractive, pairing interactions. We show that such long-ranged superconductors exhibit a new class of fluctuations in which the internal structure of the Cooper pair wave function is soft, and thus lead to "pseudogap" behavior in which the actual transition temperature is greatly depressed from its mean field value. These fluctuations are *not* phase fluctuations of the standard superconducting order parameter, and lead to a highly unusual Ginzburg-Landau description. We suggest that the crossover between the BCS limit of a short-ranged attraction and our problem is of interest in the context of superconductivity in the underdoped cuprates.

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I. INTRODUCTION

Superconductivity is a subject that has attracted prolonged interest in the condensed matter community. Much of our current understanding of this subject is based on the standard Bardeen-Cooper-Schrieffer (BCS)[1] theory, which has enjoyed great success when applied to conventional superconductors. The discovery of the "high-T_c" cuprate superconductors opened a new chapter in the study of superconductivity. It is now well understood that just like in the BCS theory, electrons in cuprate superconductors form Cooper pairs. Moreover, it has been established that the wave functions of these Cooper pairs have predominantly d-wave symmetry. However, the origin of the force that leads to Cooper pairing in these systems is not yet understood and, experimentally, significant deviations from the BCS framework (which we take to include a Fermi liquid description of the normal state) have been found near and above the transition temperature,[2] especially in the underdoped cuprates.[3] Theoretically, there is a continuum of proposals for understanding these anomalies which range from conceptually modest (if practically far reaching) enhancements of the BCS framework (spin fluctuations,[4] the crossover to pre-formed pairs,[5] etc), to the influence of various hypothesized quantum critical points[6] on to spin charge separation and non-Fermi liquid normal states.[2,7,8] Finally there are scenarios based on self-organized dimensional reduction (stripes) which are very different in spirit.[9]

This paper, also motivated by the physics of the cuprates, is firmly in the conceptually modest camp. We examine a new crossover out of the BCS corner, this time to a superconductor with long range pairing interactions, i.e. in which the range of the pairing potential L is much longer than the coherence length ($L \gg \xi$) deduced from measurements of the (anisotropic) gap in the quasiparticle spectrum. We remind the reader that in the BCS theory, the phonon-mediated, retarded attraction between electrons is often modeled as a short-range instantaneous attractive potential. As a matter of fact, it is often modeled as a constant attractive scattering potential in momentum ($\mathbf{k}$) space, which corresponds to a δ-function attraction in real space.[10,11] This is an excellent approximation because in conventional superconductors the coherence length ξ is of order ~ 1000Å, while the phonon-mediated attraction is a local (but retarded) on-site interaction with range of order the lattice spacing, and therefore the range is effectively zero compared to the size of the Cooper pairs. In such a short range model, the elementary excitations are quasiparticles with a ($\mathbf{k}$ independent) excitation gap Δ; the only collective mode below 2Δ is the linear Goldstone mode, whose energy is pushed up to the plasmon energy in real systems due to the existence of the long-range Coulomb interaction. The classic BCS limit also involves the assumption of weak coupling[12] whereupon thermally excited quasiparticles control the largely mean-field superconducting transition temperature T_c, giving rise to a universal ratio $2\Delta/T_c \approx 3.5$.

Our interest in what happens when the range of the interaction is no longer the lattice spacing but instead crosses the (suitably defined, see below) coherence length arises from four considerations.

First, ξ is much shorter in the cuprates than in conventional superconductors; in fact ξ is of order five lattice spacings in the a-b plane, and less than one lattice spacing along the c-axis. Therefore even if it turns out that the effective attractive interaction that gives rise to pairing is of the order of a lattice spacing, L and ξ will be comparable.

Second, the physics of the pseudogap regime is most simply explained by invoking a suppression of T_c by fluctuations that are small in the BCS limit. Quite generally the order parameter (or pair wavefunction) involves a relative piece and a center of mass piece, and the latter is what enters the standard Ginzburg-Landau theory. The simplest possibility is that the phase of the center of mass piece fluctuates strongly and this has been suggested in the context of the cuprates from two different standpoints, as the physics of strongly coupled pre-formed pairs[5] or that of a low superfluid density coming from a doped Mott state.[13] A more novel possibility is that the relative pair wavefunction has soft fluctuations, and as we demonstrate in this paper, these arise in the limit of a long range attraction.

Third, it appears that the model we study here is relevant to some of the proposed mechanisms for cuprate superconductivity. The first example is the inter-layer pair hopping mechanism proposed by Anderson and co-workers.[2,14] In this theory, pairing is induced by coherent *pair hopping* along the c-axis, provided that *single electron* hopping is frustrated by the non-Fermi liquid nature of the normal state. It was emphasized[14] that high T_c comes from the fact that the matrix element for pair hopping is *diagonal* in $\mathbf{k}$ space, although the symmetry of the order parameter is determined by the in-plane pairing potential. Mathematically, the pair hopping term is equivalent to an attractive potential off-diagonal in layer index, even though its physical origin is *kinetic* energy along the c-axis; in particular, being diagonal (or a δ-function) in $\mathbf{k}$ space, it corresponds to a pairing potential that has an *infinite* range in real space. A second example of this is the spin-fluctuation mechanism proposed by Pines, Scalapino, and co-workers.[4] In this theory, the range of the effective pairing interaction mediated by antiferromagnetic spin fluctuations is the spin-spin correlation length; as one reduces the doping level and approaches the antiferromagnetic instability from the overdoped side (where the system is more BCS-like), the spin-spin correlation length and therefore the range of the pairing interaction increases; it can become very large in the underdoped region, and *diverges* when approaching the antiferromagnetic instability.

Finally, given the importance of the subject of superconductivity, we feel the situation we study here (which is opposite to the familiar short-range attraction limit) is interesting in its own right. Even if it proves not to be of particular relevance to superconductivity in cuprates, it may become relevant elsewhere.[15]

Our main results may be summarized as following. We find in addition to the usual quasiparticle excitations, the system supports collective modes whose energies are significantly lower than the quasiparticle gap Δ, when $L \sim \xi$; in the limit that $L \gg \xi$, the lowest energy collective mode gap becomes *much lower* than Δ, and the number of collective modes below 2Δ becomes large. At finite temperatures, due to the thermal fluctuations of these collective modes, the transition temperature T_c becomes significantly lower than the mean field transition temperature, T_c^{MF} (which is controlled by the thermally excited quasiparticles and does not know about the collective modes). For $T_c < T < T_c^{MF}$, the electrons are paired (in momentum space) but there is no long-range phase coherence; thus the system exhibits pseudogap behavior. Also the effective Ginzburg-Landau theory needs to be modified in this case; in addition to the center of mass degree of freedom of the superconducting order parameter (or Cooper pairs), which is the only degree of freedom taken into account in the original Ginsburg-Landau description, we also need to take into account the fluctuations of the *relative* or *internal* degrees of freedom of the order parameter. As a consequence spatial gradients couple to the internal pair structure in non-trivial ways, and although we have not done the computation yet, we suspect that the critical field H_{c2} will be governed by a coherence length that is distinct from the one derived from the gap.

In the main part of this paper, we will take the attitude that this is an interesting model system to study, and work out various properties of such a model, especially those that are different from the short-range limit of the BCS theory. Discussion of the possible relevance of our considerations to the cuprates will be reserved to the Summary section.

The rest of the paper is organized as following. In Section II we revisit the Cooper problem, and keep the finite range of the attractive potential explicit. We note some problems involved in taking the limit of long-ranged potentials and specify the precise nature of the limit we consider in this paper. We find that in the limit $L \gg \xi$,[16] there are a large number of bound state solutions, whose binding energies are comparable to the ground state. These bound states correspond loosely to the collective modes in the many-body problem. In section III we study the BCS reduced Hamiltonian with a finite range attractive force, solving for both the mean-field ground state, and the zero momentum collective modes using the linearized equations of motion for the order parameter. In section IV we develop a Ginzburg-Landau effective theory from our model, using the functional integral method. We find that it is necessary to keep track of the fluctuations of both the internal and center of mass degrees of freedom of the superconducting order parameter; in particular, in the limit $L \gg \xi$, T_c is controlled by the fluctuations of the internal degrees of the order parameter, and is much lower than T_c^{MF}. In Section V we calculate the reduction of T_c from T_c^{MF} due to the thermal fluctuations of the collective modes, using the Ginzburg-Landau theory developed earlier. In Section VI we discuss the possible relevance of our work to the cuprate superconductors and mention some connections to other work in the literature.

II. THE COOPER PROBLEM

We begin by studying the Cooper problem, in which the wave function of a Cooper pair takes the form:

$$|\Psi\rangle = \sum_{0<\epsilon_k<E_c} g(\mathbf{k})c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}|\Psi_0\rangle, \tag{2.1}$$

where $|\Psi_0\rangle$ is the filled Fermi sea, $\epsilon_k = v_F(k - k_F)$ is the single electron energy measured from the Fermi level, and $E_c \ll E_F$ (Fermi energy) is an energy cutoff. In phonon mediated attraction models, E_c is usually taken to be the Debye energy. In this paper we treat E_c as a parameter that can be varied in our effective model. The Hamiltonian (for both the Cooper problem considered here and the many-body problem in the BCS reduced Hamiltonian approximation considered in the following section) takes the form

$$H = \sum_{\mathbf{k},\sigma} \epsilon_k c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{kk'}} V_{|\mathbf{k}-\mathbf{k'}|} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k'}\downarrow} c_{\mathbf{k'}\uparrow}. \tag{2.2}$$

Here

$$V_q = \frac{1}{A} \int d^2\mathbf{r} V(r) e^{-i\mathbf{q}\cdot\mathbf{r}} \tag{2.3}$$

is the Fourier transform of an attractive potential $V(r)$; A is the area of the system.[17] It is understood that the summation of the second term of Eq. (2.2) is restricted to states with $|\epsilon_k|, |\epsilon_{k'}| < E_c$. For simplicity, we assume V_q and $V(r)$ are positive definite, so the ground state for the Cooper problem has s-wave symmetry, and the many-electron system is an s-wave superconductor. We will comment on the generalization to a d-wave superconductor in the concluding section.

The Schroedinger equation that $g(\mathbf{k})$ satisfies is

$$Eg(\mathbf{k}) = 2\epsilon_k g(\mathbf{k}) - \sum_{\mathbf{k'}} V_{|\mathbf{k}-\mathbf{k'}|} g(\mathbf{k'}). \tag{2.4}$$

where E is the eigenenergy. For a bound state, we must have $E < 0$. For a general V_q, the integral equation (2.4) is difficult to solve. Cooper solved the short-range limit of this problem. In this limit, $V_q = V$ is a constant in momentum space, and therefore a δ-function in real space. It was found that there exists only one bound state solution, with s-wave symmetry; for weak coupling ($N(0)V \ll 1$, $N(0)$ being the density of states at the Fermi energy), the energy of this state is[10]

$$E \approx -2E_c e^{-2/N(0)V}. \tag{2.5}$$

This well known dependence of the binding energy on V is extremely singular. For finite but very short-range attractive potentials, there may be more bound state solutions (probably in other angular momentum channels). However, due to the singular dependence of the binding energy on the pairing potential, the binding energies of these states will be much smaller than the ground state, even if the effective pairing potential only changes slowly from one channel to another.

Here we consider the opposite limit, in which $V(r)$ has a long but finite range, L. Although the explicit form of $V(r)$ is unimportant to our basic conclusions, for concreteness we assume it has a Gaussian form:

$$V(r) = V_0 e^{-r^2/2L^2}, \tag{2.6}$$

and therefore

$$V_q = \frac{2\pi L^2}{A} V_0 e^{-q^2 L^2/2}. \tag{2.7}$$

We are interested in following the evolution of the system as L is increased and in the regime where L is large, $L \gtrsim \xi$. One possibility is to consider potentials of the "Kac type" familiar from statistical mechanics, where the range is increased while keeping the integrated strength of the potential fixed, thereby achieving an unproblematic thermodynamic limit. A second possibility, perhaps more appropriate to attractive interactions generated by the system itself,[18] is to keep the interaction of fixed magnitude (V_0) but to restrict its operation to an increasingly

narrow shell around the Fermi surface. We will choose the second course here. An easy estimate shows that we need to pick the cutoff energy

$$E_c \propto L^{-D} \tag{2.8}$$

in order to keep the potential energy per particle finite. This also has the advantage that, by construction, is suppresses the tendency to phase separation that would otherwise be a complication with attractive long range interactions.[19] There is however, a final caveat. For technical reasons, starting in the next paragraph, we will often employ a gradient expansion in momentum space. This is not quite consistent with the cutoff procedure but for the results of interest we will find that the precise value of the cutoff will enter weakly, leading us to believe that a better set of calculations will not alter our general scenario.

Since $V_{|\mathbf{k}-\mathbf{k}'|}$ goes to zero rapidly for $|\mathbf{k}-\mathbf{k}'| > 1/L$, if $g(\mathbf{k})$ varies slowly on the scale of $1/L$, we may perform a gradient expansion for $g(\mathbf{k})$ in $\mathbf{k}$-space, in the last term of Eq. (2.4):

$$\begin{aligned}
\sum_{\mathbf{k}'} V_{|\mathbf{k}-\mathbf{k}'|} g(\mathbf{k}') &= \sum_{\mathbf{k}'} V_{|\mathbf{k}-\mathbf{k}'|} \left[g(\mathbf{k}) + \nabla_{\mathbf{k}} g(\mathbf{k}) \cdot (\mathbf{k}'-\mathbf{k}) + \frac{1}{2}\sum_{\mu\nu} \frac{\partial^2 g(\mathbf{k})}{\partial k_\mu \partial k_\nu}(k'_\mu - k_\mu)(k'_\nu - k_\nu) + \cdots \right] \\
&\approx \left(\sum_{\mathbf{k}'} V_{|\mathbf{k}-\mathbf{k}'|} \right) g(\mathbf{k}) + \frac{1}{4} \left(\sum_{\mathbf{k}'} |\mathbf{k}-\mathbf{k}'|^2 V_{|\mathbf{k}-\mathbf{k}'|} \right) \nabla_{\mathbf{k}}^2 g(\mathbf{k}) \\
&= V_0 g(\mathbf{k}) + \frac{V_0}{2L^2} \nabla_{\mathbf{k}}^2 g(\mathbf{k}),
\end{aligned} \tag{2.9}$$

in which we have neglected higher gradient terms. Thus within the gradient expansion, Eq. (2.4) reduces to

$$-\frac{1}{2M}\nabla_{\mathbf{k}}^2 g(\mathbf{k}) + (2\epsilon_k - V_0)g(\mathbf{k}) = E g(\mathbf{k}), \tag{2.10}$$

where $M = L^2/V_0$. This *differential* equation (2.10) is identical to the Schroedinger equation of a particle confined in an annulus (with some unknown, but calculable, boundary condition) with inner and outer radii k_F and $k_F + E_c/v_F$ respectively, experiencing a "potential" $U(k) = -V_0 + 2\epsilon_k = -V_0 + 2v_F(k - k_F)$. It is interesting to note that in Eq. (2.10), the "kinetic energy" term comes from the two-body interaction in the original Hamiltonian, while the "potential" term actually is the original kinetic energy term.

For large L, the "mass" M is large, and we may use WKB approximation to analyze Eq. (2.10). The following conclusions follow straightforwardly:

i) The ground state is in the s-wave channel, with energy

$$E_0 \approx -V_0 + c\left(\frac{v_F\sqrt{V_0}}{L}\right)^{2/3} \approx -V_0. \tag{2.11}$$

Here c is a constant of order 1; it is approximately $\left(\frac{3\pi}{2\sqrt{2}}\right)^{2/3}$ for hard wall boundary conditions. We find the binding energy is essentially the depth of the attractive two-body potential V_0; this dependence is much less singular than the short-range limit defined above. Also the cutoff E_c does not enter explicitly.[20] At this point we define a "coherence length"

$$\xi = 2v_F/\pi|E_0| \approx 2v_F/\pi V_0, \tag{2.12}$$

in analogy to the BCS coherence length at zero temperature:[10,11] $\xi_0 = v_F/(\pi\Delta(0))$, where $\Delta(0)$ is the quasiparticle gap at zero temperature. What we have in mind is that the binding energy in the Cooper problem will be the same as twice the quasiparticle gap in the BCS problem. This is *not* the case in short-range models; however, as we will see later, it is indeed true in the present case. We need to emphasize however, that ξ is *not* the size of the Cooper pair wave function ℓ in the two-body problem studied in this section (this is the case even in short-range models); the latter can be estimated easily from Eq. (2.10). The size of the ground state wave function in momentum space for the Hamiltonian (2.10) is

$$\Delta k \sim \left(\frac{1}{v_F M}\right)^{1/3} = \left(\frac{V_0}{v_F L^2}\right)^{1/3}, \tag{2.13}$$

thus $\ell \sim 1/\Delta k \sim (v_F L^2/V_0)^{1/3}$; it actually increases with L; however the sublinear dependence means for large L we do have $\ell \ll L$. This points to some ambiguities in what is meant by the coherence length in our problem and

presumably spatial gradients may be governed by a different coherence length than the gap coherence length. We note that this possibility has been raised on completely different grounds in Ref. 7.

In order for the gradient expansion performed above to be valid, we must have $\Delta k \gtrsim 1/L$, i.e., $g(\mathbf{k})$ of the ground state varying slowly over the scale $1/L$, which is the range of V_q. This leads to the condition

$$L \gtrsim 2v_F/\pi V_0 \approx \xi, \tag{2.14}$$

as advertised earlier. It turns out that throughout this paper the gradient expansion in momentum space is the key technique that allows explicit calculations of various physical quantities to be made, and the above condition defines the range of its validity. For a given potential with fixed V_0 and L, one may also interpolate between the short-range and long-range limits by varying v_F. In the following we assume Eq. (2.14) is satisfied, and pay particular attention to the limit $L \gg \xi$.

ii) The number of bound states in the s-wave ($l = 0$) channel is $N_0 \approx \frac{\sqrt{2}L}{3\xi}$.

iii) The largest angular momentum channel that supports a bound state has angular momentum $l_{max} \approx \sqrt{2}k_F L$.

iv) The total number of bound states is $N_{tot} = \sum_l N_l \approx \frac{\pi}{6} k_F L^2/\xi$.

v) The energy spacing between the ground state and the first excited state is $\Delta E \approx \frac{V_0}{2k_F^2 L^2} \ll |E_0|$, i.e., it is much lower than the binding energy of the ground state itself, in contrast to the short range case.

The gradient expansion does not apply to all bound states, even if $L \gg \xi$. One can show that it only applies to states with energy measured from the ground state (or bottom of the potential well) $\Delta E \lesssim V_0 \xi^2/L^2$. Nevertheless the estimate of number of bound states in various channels should be qualitatively correct.

These large number of low-energy bound states correspond loosely to the low-energy collective modes in the many body problem studied in the following sections, and they lead to a significant reduction of T_c from its mean-field value T_c^{MF} (whose scale is set by the gap which is the same as the Cooper pair binding energy here, $T_c^{MF} \sim \Delta \sim V_0$). This may be understood heuristically in the following way. When the temperature T is reduced to T_c^{MF}, electrons start to form Cooper pairs. In the short-range BCS model, there is only one way to form a Cooper pair, thus all pairs condense into one state immediately and long-range coherence is established. In the model studied here however, there are many different ways to form Cooper pairs and their binding energies are comparable; thus the system needs to go to much lower temperature for all the Cooper pairs to condense into the ground state, hence $T_c \ll T_c^{MF}$. For $T_c < T < T_c^{MF}$, the electrons are paired (and therefore gapped), but there is no long-range superconducting coherence, and the system exhibit pseudogap behavior.

III. MEAN FIELD THEORY AND COLLECTIVE MODES IN THE BCS REDUCED HAMILTONIAN

In this section we study the BCS reduced Hamiltonian, Eq. (2.2). Instead of solving the two-body problem in the above section, here we study the many-body problem. The difference between Eq. (2.2) and the full many-body problem is that in Eq. (2.2), only pairs of electrons with opposite spin and total momentum zero scatter each other. (we return to the full problem, i.e. when the sum over the interacting momenta is constrained only by momentum conservation in the next section.) This leads to a very special property, that an unpaired electron never gets scattered, and thus the number of unpaired electron (0 or 1) for any $\mathbf{k}$ and spin orientation is a conserved quantity. The ground state for this Hamiltonian is in the subspace in which no such unpaired electron exists, namely the pair of single electron states $\mathbf{k} \uparrow$ and $-\mathbf{k} \downarrow$ are either both occupied or both empty. In this subspace, one may map the problem onto a quantum pseudospin problem.[21] We introduce a pseudospin-1/2 operator for each $\mathbf{k}$:

$$S_\mathbf{k}^z = (c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\uparrow} + c_{-\mathbf{k}\downarrow}^\dagger c_{-\mathbf{k}\downarrow} - 1)/2,$$
$$S_\mathbf{k}^+ = c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger,$$
$$S_\mathbf{k}^- = c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}. \tag{3.1}$$

In this mapping, the pseudospin is pointing up when a pair of single electron states are occupied, and down when they are empty; pair creation/annihilation operators map onto spin-flip operators. The Hamiltonian now takes the form (up to a constant):

$$H = \sum_\mathbf{k} 2\epsilon_\mathbf{k} S_\mathbf{k}^z - \frac{1}{2} \sum_{\mathbf{k}\mathbf{k}'} V_{|\mathbf{k}-\mathbf{k}'|}(S_\mathbf{k}^+ S_{\mathbf{k}'}^- + S_{\mathbf{k}'}^+ S_\mathbf{k}^-) = \sum_\mathbf{k} 2\epsilon_\mathbf{k} S_\mathbf{k}^z - \sum_{\mathbf{k}\mathbf{k}'} V_{|\mathbf{k}-\mathbf{k}'|}(S_\mathbf{k}^x S_{\mathbf{k}'}^x + S_{\mathbf{k}'}^y S_\mathbf{k}^y), \tag{3.2}$$

i.e., the kinetic energy maps onto a Zeeman field that depends linearly on $\epsilon_\mathbf{k}$ (and changes sign at the Fermi surface), which couples to the z component of the pseudospins; the pairing interaction maps onto a *ferromagnetic* coupling among the xy components of the pseudospins.

In the mean field approximation, one replaces the ferromagnetic coupling among the xy components of the spins by an average field:

$$H_{MF} = -\sum_{\mathbf{k}} \mathbf{B}_{\mathbf{k}}^0 \cdot \mathbf{S}_{\mathbf{k}}, \tag{3.3}$$

where $\mathbf{B}_{\mathbf{k}}^0$ satisfies the self-consistency equation:

$$\mathbf{B}_{\mathbf{k}}^0 = -2\epsilon_{\mathbf{k}}\hat{z} + 2\sum_{\mathbf{k}'} V_{|\mathbf{k}-\mathbf{k}'|}\langle \mathbf{S}_{\mathbf{k}'}^{\perp}\rangle. \tag{3.4}$$

Here $\mathbf{S}_{\mathbf{k}}^{\perp}$ stands for the xy components of $\mathbf{S}_{\mathbf{k}}$. In the ground state, $\mathbf{S}_{\mathbf{k}}$ points in the direction of $\mathbf{B}_{\mathbf{k}}^0$; the elementary excitations are single pseudospin flips, which corresponds to a pair of Bogliubov quasiparticles with opposite momenta and (real) spin.[22]

In the short-range limit $L \to 0$, the range of $V_{|\mathbf{k}-\mathbf{k}'|}$, $1/L$, *diverges* in $\mathbf{k}$ space. Thus all the pseudospins are equally coupled, no matter how far away they are in $\mathbf{k}$ space. In this limit, the mean field approximation becomes exact; the only collective mode is a zero mode corresponds to the global rotation of all the pseudospins along the z-axis, reflecting the broken XY symmetry of the ground state.[21] All other excitations in this subspace may be described by pseudospin flips, or pairs of quasiparticles.

The situation, however, becomes quite different, when L becomes large. In this case $V_{|\mathbf{k}-\mathbf{k}'|}$ becomes *short-ranged* in $\mathbf{k}$ space; one may actually divide the $\mathbf{k}$ space into blocks of size $\frac{1}{L} \times \frac{1}{L}$; within each block all the spins are strongly coupled and are effectively locked into a very big single spin; for different blocks, however, couplings are restricted to neighboring blocks. In such an XY ferromagnet with "short range" couplings among the blocked spins, there are (pseudo)spin-wave like collective excitations in addition to single pseudospin flips, whose energies can become significantly lower than pseudospin flips for large L, as the number of blocks increases as $\sim L^2$.

We begin by analyzing the mean field equation, Eq. (3.4), at zero temperature. Without losing generality, we may assume $\mathbf{B}_{\mathbf{k}}^0$ is in the $x-z$ plane:

$$\mathbf{B}_{\mathbf{k}}^0 = -2\epsilon_{\mathbf{k}}\hat{z} + 2\Delta_{\mathbf{k}}\hat{x}, \tag{3.5}$$

and the self-consistency equation for $\Delta_{\mathbf{k}}$ is

$$\Delta_{\mathbf{k}} = \sum_{\mathbf{k}'} V_{|\mathbf{k}-\mathbf{k}'|}\langle \mathbf{S}_{\mathbf{k}}^x\rangle = \frac{1}{2}\sum_{\mathbf{k}'} \frac{V_{|\mathbf{k}-\mathbf{k}'|}\Delta_{\mathbf{k}'}}{\sqrt{\epsilon_{\mathbf{k}'}^2 + \Delta_{\mathbf{k}'}^2}}, \tag{3.6}$$

which is the familiar BCS gap equation.[10,11] In the limit $L \to 0$, $V_{|\mathbf{k}-\mathbf{k}'|} = V$ becomes independent of $|\mathbf{k}-\mathbf{k}'|$, so does $\Delta_{\mathbf{k}} = \Delta$, and Eq. (3.6) reduces to

$$1 = \frac{1}{2}\sum_{\mathbf{k}'} \frac{V}{\sqrt{\epsilon_{\mathbf{k}'}^2 + \Delta^2}}. \tag{3.7}$$

In the weak coupling limit, its solution is $\Delta = 2E_c e^{-1/N(0)V}$. This gap is much larger than the binding energy in the Cooper problem, Eq. (2.5), due to the factor of two difference in the exponential.

The situation becomes very different in the opposite limit, that L becomes large. In this case $V_{|\mathbf{k}-\mathbf{k}'|}$ becomes short-ranged in $\mathbf{k}$ space; thus on the right hand side of Eq. (3.6), we may replace $\Delta_{\mathbf{k}'}$ and $\epsilon_{\mathbf{k}'}$ by $\Delta_{\mathbf{k}}$ and $\epsilon_{\mathbf{k}}$ to first approximation, provided that they vary slowly on the scale of $1/L$. In this approximation one finds in the limit $L \to \infty$,

$$\Delta_{\mathbf{k}} = \sqrt{\frac{V_0^2}{4} - \epsilon_{\mathbf{k}}^2}, \tag{3.8}$$

for $|\epsilon_{\mathbf{k}}| < V_0/2$, and $\Delta_{\mathbf{k}} = 0$ otherwise. For large but finite L, $\Delta_{\mathbf{k}}$ is nonzero but exponentially small for $|\epsilon_{\mathbf{k}}| > V_0/2$. Thus $\Delta_{\mathbf{k}}$ is $\mathbf{k}$ *dependent*, and reaches its maximum for $k = k_F$, in which case

$$\Delta_{k_F} = \frac{V_0}{2}. \tag{3.9}$$

We find the quasiparticle gap and the binding energy are indeed set by the same energy scale in the Cooper problem, as advertised earlier; in particular, the gap for creating a pair of quasiparticles on the Fermi surface, $2\Delta_{k_F} = V_0$,

is exactly the Cooper pair binding energy, in the limit $L \to \infty$. Following standard convention,[10] we introduce the coherence length

$$\xi = \frac{v_F}{\pi \Delta_{k_F}} \approx \frac{2v_F}{\pi V_0}. \tag{3.10}$$

This definition indeed matches that of ξ in the Cooper problem Eq. 2.12, in the limit that L is large. And one can easily show that the large L approximation is valid for

$$L \gtrsim \xi. \tag{3.11}$$

Within the mean field theory, the elementary excitations are the Bogliubov quasiparticles, with the standard spectrum

$$E_k = \sqrt{\epsilon_k^2 + \Delta_k^2}. \tag{3.12}$$

At finite temperature, the self-consistent mean field equation becomes

$$\Delta_{\mathbf{k}} = \frac{1}{2} \sum_{\mathbf{k}'} \frac{V_{|\mathbf{k}-\mathbf{k}'|} \Delta_{\mathbf{k}'}}{E_{\mathbf{k}}} \tanh \frac{\beta E_{\mathbf{k}}}{2}, \tag{3.13}$$

where $\beta = \frac{1}{T}$ is the inverse temperature (we set the Boltzmann constant k_B to be 1). We can use the above equation to determine the mean-field transition temperature T_c^{MF}. In the $L \to \infty$ limit, it is particularly simple; in this case Eq. (3.13) reduces to

$$1 = \frac{V_0}{2E_{\mathbf{k}}} \tanh \frac{\beta E_{\mathbf{k}}}{2}. \tag{3.14}$$

Using the facts that as $T \to T_c^{MF}$, $\Delta_{\mathbf{k}} \to 0$ and $E_{\mathbf{k}} \to \epsilon_{\mathbf{k}}$, and focusing on $|\mathbf{k}| \sim k_F$ where the pairing instability is the strongest, we find

$$T_c^{MF} = \frac{V_0}{4} \tag{3.15}$$

in this limit. Thus the ratio $2\Delta_{k_F}/T_c^{MF} = 4$, which is slightly bigger than the BCS ratio of 3.5.

There are a couple of new features of the mean-field solution in the large L regime, that deserve some further discussion. (i) Unlike the case of short-range attraction originally considered by BCS, where the gap Δ is a constant in momentum space at all temperatures (within the cutoff where the pairing interaction is nonzero), here $\Delta_{\mathbf{k}}$ has strong momentum dependence, being maximum at the Fermi surface and decreasing as one moves away from it. At $T = 0$, the momentum dependence Eq. (3.8) is such that the quasiparticle dispersion is essentially *flat* near the Fermi surface, for sufficiently large L. (ii) The temperature T not only affects the overall scale of the gap, but also its momentum dependence. Taken at face value, one would conclude from Eq. (3.14) that T_c^{MF} depends on $\epsilon_{\mathbf{k}}$ in the following way:

$$T_c^{MF}(\epsilon_{\mathbf{k}}) = T_c^{MF} \frac{2\epsilon_{\mathbf{k}}/V_0}{\tanh^{-1}(2\epsilon_{\mathbf{k}}/V_0)} \leq T_c^{MF}. \tag{3.16}$$

What this really means, of course, is that the gap $\Delta_{\mathbf{k}}$ remains exponentially small for $T_c^{MF}(\epsilon_{\mathbf{k}}) \lesssim T < T_c^{MF}$. Thus in some sense T_c^{MF} is a "local" property in the momentum space, increasing monotonically, and scaling roughly with the size of local gap $\Delta_{\mathbf{k}}(T = 0)$ unless $\Delta_{\mathbf{k}}(T = 0)$ is very small.

In the following we go beyond mean field theory and study the collective excitations of the system at $T = 0$. To do that, we study the equations of motion for $\mathbf{S_k}$[21]:

$$\frac{d\mathbf{S_k}}{dt} = -i[\mathbf{S_k}, H] = \mathbf{S_k} \times \mathbf{B_k}, \tag{3.17}$$

where

$$\mathbf{B_k} = -2\epsilon_{\mathbf{k}} \hat{z} + 2 \sum_{\mathbf{k}'} V_{|\mathbf{k}-\mathbf{k}'|} \mathbf{S}_{\mathbf{k}'}^{\perp}. \tag{3.18}$$

Write $\mathbf{B_k} = \mathbf{B_k^0} + \delta\mathbf{B_k}$ and $\mathbf{S_k} = \mathbf{S_k^0} + \delta\mathbf{S_k}$, where $\mathbf{S_k^0}$ is the expectation value of $\mathbf{S_k}$ in the mean field ground state, and linearizing the equation of motion by neglecting terms proportional to $\delta\mathbf{S_k} \times \delta\mathbf{B_k}$, we obtain

$$\frac{d\delta\mathbf{S_k}}{dt} = \delta\mathbf{S_k} \times \mathbf{B_k^0} + \mathbf{S_k^0} \times \delta\mathbf{B_k}. \tag{3.19}$$

$\delta\mathbf{S_k}$ should be perpendicular to $\mathbf{S_k^0}$. If $\mathbf{S_k^0}$ is in the $x - z$ plane, we may assume

$$\delta\mathbf{S_k} = \delta S_k^y \hat{y} + \delta S_k^\| \hat{e}_\mathbf{k}, \tag{3.20}$$

where $\hat{e}_\mathbf{k}$ is in the $x - z$ plane, but perpendicular to $\mathbf{S_k^0}$. Thus Eq. (3.19) reduces to

$$\frac{d\delta S_k^y}{dt} = \frac{1}{2}\delta B_\mathbf{k}^x \cos\theta_\mathbf{k} - B_\mathbf{k}^0 \delta S_\mathbf{k}^\|, \tag{3.21}$$

$$\frac{d\delta S_k^\|}{dt} = B_\mathbf{k}^0 \delta S_\mathbf{k}^y - \frac{1}{2}\delta B_\mathbf{k}^y. \tag{3.22}$$

Here $\theta_\mathbf{k}$ is the angle between $\mathbf{B_k^0}$ and the $\hat{z}$ direction, and

$$\delta B_\mathbf{k}^x = 2\sum_{\mathbf{k'}} V_{|\mathbf{k}-\mathbf{k'}|}\delta S_{\mathbf{k'}}^\| \cos\theta_{\mathbf{k'}}, \tag{3.23}$$

$$\delta B_\mathbf{k}^y = 2\sum_{\mathbf{k'}} V_{|\mathbf{k}-\mathbf{k'}|}\delta S_{\mathbf{k'}}^y. \tag{3.24}$$

Thus for a mode with frequency ω and $\delta S_k^y \propto \phi_\mathbf{k}$, it must satisfy

$$-\omega^2 \phi_\mathbf{k} = -(B_\mathbf{k}^0)^2 \phi_\mathbf{k} + B_\mathbf{k}^0 \sum_{\mathbf{k'}} V_{|\mathbf{k}-\mathbf{k'}|}\phi_{\mathbf{k'}} + \cos\theta_\mathbf{k} \sum_{\mathbf{k'}} V_{|\mathbf{k}-\mathbf{k'}|} B_{\mathbf{k'}}^0 \cos\theta_{\mathbf{k'}}\phi_{\mathbf{k'}}$$

$$- \cos\theta_\mathbf{k} \sum_{\mathbf{k'k''}} V_{|\mathbf{k}-\mathbf{k'}|} V_{|\mathbf{k'}-\mathbf{k''}|}\cos\theta_{\mathbf{k'}}\phi_{\mathbf{k''}}. \tag{3.25}$$

Performing a gradient expansion similar in spirit to the previous section, we obtain

$$\omega^2 \phi_\mathbf{k} = \tilde{U}_\mathbf{k}\phi_\mathbf{k} - \frac{1}{4}B_\mathbf{k}^0 \sum_{\mathbf{k'}}|\mathbf{k}-\mathbf{k'}|^2 V_{|\mathbf{k}-\mathbf{k'}|}\nabla_\mathbf{k}^2 \phi_\mathbf{k} - \frac{\cos\theta_\mathbf{k}}{4}(\sum_{\mathbf{k'}}|\mathbf{k}-\mathbf{k'}|^2 V_{|\mathbf{k}-\mathbf{k'}|})\nabla_\mathbf{k}^2(\cos\theta_\mathbf{k}B_\mathbf{k}^0\phi_\mathbf{k})$$

$$+ \cos\theta_\mathbf{k}[\sum_{\mathbf{k'k''}} V_{|\mathbf{k}-\mathbf{k'}|} V_{|\mathbf{k'}-\mathbf{k''}|}\cos\theta_{\mathbf{k'}}(\mathbf{k''} - \mathbf{k})] \cdot \nabla_\mathbf{k}\phi_\mathbf{k}$$

$$+ \frac{1}{2}\cos\theta_\mathbf{k}[\sum_{\mathbf{k'k''}} V_{|\mathbf{k}-\mathbf{k'}|} V_{|\mathbf{k'}-\mathbf{k''}|}\cos\theta_{\mathbf{k'}}(k_\mu'' - k_\mu)(k_\nu'' - k_\nu)]\partial_\mu\partial_\nu\phi_\mathbf{k} + \cdots, \tag{3.26}$$

where the ellipses stand for terms that involve higher gradients which we neglect. Here

$$\tilde{U}_\mathbf{k} = (B_\mathbf{k}^0)^2 - B_\mathbf{k}^0 \sum_{\mathbf{k'}} V_{|\mathbf{k}-\mathbf{k'}|} - \cos^2\theta_\mathbf{k}(B_\mathbf{k}^0 \sum_{\mathbf{k'}} V_{|\mathbf{k}-\mathbf{k'}|} - \sum_{\mathbf{k'k''}} V_{|\mathbf{k}-\mathbf{k'}|} V_{|\mathbf{k'}-\mathbf{k''}|}) - \frac{V_0^2}{2L^2}\nabla_\mathbf{k}^2\cos\theta_k$$

$$= (B_\mathbf{k}^0 - \cos^2\theta_\mathbf{k}V_0)(B_\mathbf{k}^0 - V_0) - \frac{V_0^2}{2L^2}\nabla_\mathbf{k}^2\cos\theta_k. \tag{3.27}$$

In the limit that L becomes very large, one finds $\tilde{U}_\mathbf{k} = 0$ for $|\epsilon_\mathbf{k}| < V_0$, and $\tilde{U}_\mathbf{k} = (\epsilon_\mathbf{k} - V_0)^2$ otherwise.

For low energy modes, we expect $\phi_\mathbf{k}$ to be centered near $k = k_F$. Thus in a somewhat crude approximation, we may neglect the gradient terms in Eq. (3.26) proportional to $\cos\theta_\mathbf{k}$, as $\cos\theta_\mathbf{k} \to 0$ as $k \to k_F$; and assume $\tilde{U}_\mathbf{k}$ to become infinite for $|\epsilon_\mathbf{k}| > V_0$. Within this approximation (which should give qualitatively correct results), we obtain the collective mode frequencies to be

$$\omega_{mn} = \frac{V_0}{\sqrt{2}}\sqrt{\frac{m^2}{k_F^2 L^2} + \frac{\pi^4 n^2}{(L/\xi)^2}}, \tag{3.28}$$

where m and n are "momentum"-like quantum numbers along and perpendicular to the Fermi surface respectively. So the mode frequency is linear in these "momenta", as one would expect for the spin wave spectrum in an XY ferromagnet. It is clear that the energies of these collective modes become much lower than those of single pseudospin flips (corresponding to quasiparticle excitations), when L is large. Since our discussion in this section is restricted to the BCS reduced Hamiltonian in which only pairs with zero total momentum interact with each other, these modes are exciton-like collective modes at zero momentum; they will acquire the usual dispersion of exciton modes at finite momentum. The term "exciton" indicates that the zero momentum excitation consists of a pair of quasiparticles bound into a pair state *different* from that of the condensate which is why the state lies in the gap between the ground state and the two quasiparticle continuum. The availability of such states (as we already saw in the previous section) is due to the long range of the interaction.

IV. GINZBURG-LANDAU THEORY AND ELECTROMAGNETIC RESPONSE

In this section we use the functional integral formalism to derive the effective Ginzburg-Landau theory near their T_c^{MF} for superconductors with finite range attractive interactions, and use it to derive their analog of the London equation.

Let us consider the Hamiltonian

$$\hat{H} = \hat{T} + \hat{V} = \sum_{k\sigma}(\epsilon_k - \mu)c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} - \int d\mathbf{x}d\mathbf{x}' V(|\mathbf{x} - \mathbf{x}'|)\Psi_\uparrow^\dagger(\mathbf{x})\Psi_\downarrow^\dagger(\mathbf{x}')\Psi_\downarrow(\mathbf{x}')\Psi_\uparrow(\mathbf{x}). \tag{4.1}$$

Here $V(x) \geq 0$ represents an attractive interaction. We will primarily be interested in singlet pairing in the ground state; so we neglect interactions between electrons with the same spin.[23] While the interaction is written in a pairwise form, it is understood that a cutoff exists in momentum space so that only electrons that are close enough to the Fermi surface interact with each other, in the same manner as in previous sections. The difference here is that the interaction is no longer restricted to pairs of electrons with zero total momentum; thus the above Hamiltonian represents the full many-body problem, albeit with the parallel spin interaction ignored.

One may also describe the system using an Euclidean action in terms of Grassman variables:

$$S[\Psi, \overline{\Psi}] = S_0[\Psi, \overline{\Psi}] - \int_0^\beta d\tau \int d\mathbf{x}d\mathbf{x}' V(|\mathbf{x} - \mathbf{x}'|)\overline{\Psi}_\uparrow(\mathbf{x}, \tau)\overline{\Psi}_\downarrow(\mathbf{x}', \tau)\Psi_\downarrow(\mathbf{x}', \tau)\Psi_\uparrow(\mathbf{x}, \tau), \tag{4.2}$$

where S_0 is the action for free electrons, and τ is the imaginary time. The partition function is

$$Z = \int D\overline{\Psi}D\Psi e^{-S[\Psi, \overline{\Psi}]}. \tag{4.3}$$

We now decouple the quartic term in S by introducing a pair of Hubbard-Stratonovich fields $\Delta(\mathbf{x}, \mathbf{x}', \tau)$ and $\overline{\Delta}(\mathbf{x}, \mathbf{x}', \tau)$, which will become the superconducting order parameter in the sequel:

$$S[\Psi, \overline{\Psi}, \Delta, \overline{\Delta}] = S_0 - \int_0^\beta d\tau \int d\mathbf{x}d\mathbf{x}'[\Delta(\mathbf{x}, \mathbf{x}', \tau)\overline{\Psi}_\uparrow(\mathbf{x}, \tau)\overline{\Psi}_\downarrow(\mathbf{x}', \tau) + \overline{\Delta}(\mathbf{x}, \mathbf{x}', \tau)\Psi_\downarrow(\mathbf{x}', \tau)\Psi_\uparrow(\mathbf{x}, \tau)]$$
$$+ \int_0^\beta d\tau \int d\mathbf{x}d\mathbf{x}' \frac{|\Delta(\mathbf{x}, \mathbf{x}', \tau)|^2}{V(|\mathbf{x} - \mathbf{x}'|)}. \tag{4.4}$$

With this decoupling, the fermionic action becomes quadratic, and can be integrated out, after which we obtain an effective action in terms of the order parameter $\Delta(\mathbf{x}, \mathbf{x}', \tau)$:

$$S_e[\Delta, \overline{\Delta}] = \int_0^\beta d\tau d\mathbf{x}d\mathbf{x}' \frac{|\Delta(\mathbf{x}, \mathbf{x}', \tau)|^2}{V(|\mathbf{x} - \mathbf{x}'|)} - \log Z[\Delta, \overline{\Delta}], \tag{4.5}$$

where

$$Z[\Delta, \overline{\Delta}] = \int D\overline{\Psi}D\Psi e^{-S_0[\Psi, \overline{\Psi}] + \int_0^\beta d\tau d\mathbf{x}d\mathbf{x}'[\Delta(\mathbf{x}, \mathbf{x}', \tau)\overline{\Psi}_\uparrow(\mathbf{x}, \tau)\overline{\Psi}_\downarrow(\mathbf{x}', \tau) + \overline{\Delta}(\mathbf{x}, \mathbf{x}', \tau)\Psi_\downarrow(\mathbf{x}', \tau)\Psi_\uparrow(\mathbf{x}, \tau)]}. \tag{4.6}$$

The mean-field solution corresponds to the saddle point of $S_e[\Delta, \overline{\Delta}]$:

$$\frac{\delta S_e[\Delta, \overline{\Delta}]}{\delta \overline{\Delta}(\mathbf{x}, \mathbf{x}')}\Big|_{\Delta=\Delta_s} = \frac{\Delta_s(\mathbf{x}, \mathbf{x}')}{V(\mathbf{x} - \mathbf{x}')} - \frac{\delta \log Z[\Delta, \overline{\Delta}]}{\delta \overline{\Delta}(\mathbf{x}, \mathbf{x}')}\Big|_{\Delta=\Delta_s} = \frac{\Delta_s(\mathbf{x}, \mathbf{x}')}{V(\mathbf{x} - \mathbf{x}')} - \langle \Psi_\downarrow(\mathbf{x}')\Psi_\uparrow(\mathbf{x})\rangle_{\Delta_s} = 0, \tag{4.7}$$

where $\langle\rangle_{\Delta_s}$ stands for quantum and thermal averaging in the presence of the pairing field $\Delta_s(\mathbf{x}, \mathbf{x}')$. Here we have assumed a static saddle point so that Δ_s has no τ dependence. Further assuming that at the saddle point, $\Delta_s(\mathbf{x}, \mathbf{x}') = \Delta_s(\mathbf{x} - \mathbf{x}')$ is translationally invariant, it is easy to show that Eq. (4.7) is equivalent to Eq. (3.6), upon the identification

$$\Delta_{\mathbf{k}} = \frac{1}{A}\int d\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}}\Delta_s(\mathbf{x}). \tag{4.8}$$

The functional integral formalism can be used to derive the effective Ginzburg-Landau free energy, in the vicinity of T_c^{MF}. This has been done for the short-range attractive interactions.[24] Here we use it to derive the appropriate Ginzburg-Landau free energy for finite range attractive interactions.

Our starting point is the effective action, Eq. (4.5). Near T_c^{MF}, we may make two simplifications: i) We may neglect the τ dependence of Δ as we expect the thermal fluctuations to dominate the quantum fluctuations; ii) We may expand S_e in powers of Δ. The quadratic terms take the form

$$S_e^{(2)}[\Delta, \overline{\Delta}] = \beta \int d\mathbf{x}d\mathbf{y}\frac{|\Delta(\mathbf{x}, \mathbf{y})|^2}{V(|\mathbf{x} - \mathbf{y}|)} - \int d\mathbf{x}_1 d\mathbf{y}_1 d\mathbf{x}_2 d\mathbf{y}_2 Q(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)\Delta(\mathbf{x}_1, \mathbf{y}_1)\overline{\Delta}(\mathbf{x}_2, \mathbf{y}_2), \tag{4.9}$$

where

$$
\begin{aligned}
Q(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2) &= \frac{\delta^2 \log Z}{\delta \Delta(\mathbf{x}_1, \mathbf{y}_1)\delta \overline{\Delta}(\mathbf{x}_2, \mathbf{y}_2)}\Big|_{\Delta=0} \\
&= \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \langle \Psi_\downarrow(\mathbf{y}_2, \tau_2)\Psi_\uparrow(\mathbf{x}_2, \tau_2)\overline{\Psi}_\uparrow(\mathbf{x}_1, \tau_1)\overline{\Psi}_\downarrow(\mathbf{y}_1, \tau_1)\rangle_c \\
&= \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 G_0(\mathbf{y}_2 - \mathbf{y}_1; \tau_2 - \tau_1)G_0(\mathbf{x}_2 - \mathbf{x}_1; \tau_2 - \tau_1) \\
&= \sum_{i\omega_n} G_0(\mathbf{y}_2 - \mathbf{y}_1; i\omega_n)G_0(\mathbf{x}_2 - \mathbf{x}_1; -i\omega_n). \tag{4.10}
\end{aligned}
$$

Here G_0 is the *normal state* (non-interacting) single electron Green's function, ω_n's are fermion Masubara frequencies, and $\langle\rangle_c$ stands for connected contractions in the average.

We now introduce "center of mass" and "relative" coordinates for the order parameter $\Delta(\mathbf{x}, \mathbf{y})$:

$$\mathbf{R} = (\mathbf{x} + \mathbf{y})/2, \qquad \mathbf{r} = \mathbf{y} - \mathbf{x}, \tag{4.11}$$

and Fourier transform with respect to the relative coordinate $\mathbf{r}$:

$$\Delta(\mathbf{R}, \mathbf{k}) = \int d\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}}\Delta(\mathbf{R} - \mathbf{r}/2, \mathbf{R} + \mathbf{r}/2). \tag{4.12}$$

In a uniform ($\mathbf{R}$ independent) configuration, $\Delta(\mathbf{R}, \mathbf{k})$ becomes $\Delta_{\mathbf{k}}$.

In terms of $\Delta(\mathbf{R}, \mathbf{k})$, the first term of Eq. (4.9) becomes

$$
\begin{aligned}
\beta \int d\mathbf{x}d\mathbf{y}\frac{|\Delta(\mathbf{x}, \mathbf{y})|^2}{V(|\mathbf{x} - \mathbf{y}|)} &= \beta \int d\mathbf{R}\int d\mathbf{r}\frac{|\Delta(\mathbf{R} - \frac{\mathbf{r}}{2}, \mathbf{R} + \frac{\mathbf{r}}{2})|^2}{V(r)} \\
&= \frac{\beta}{A^2}\int d\mathbf{R}\sum_{\mathbf{k}_1\mathbf{k}_2}\Delta(\mathbf{R}, \mathbf{k}_1)\overline{\Delta}(\mathbf{R}, \mathbf{k}_2)F(|\mathbf{k}_1 - \mathbf{k}_2|), \tag{4.13}
\end{aligned}
$$

where

$$F(|\mathbf{k}_1 - \mathbf{k}_2|) = \int d\mathbf{r} e^{i(\mathbf{k}_1 - \mathbf{k}_2)\cdot\mathbf{r}}/V(r). \tag{4.14}$$

This expression is problematic as it stands, for $1/V(r)$ does not, in a strict sense, possess a Fourier transform in a truly infinite system. However, it can be defined in a large but finite-size system with periodic boundary conditions; and as we will see in a few lines, this is only an intermediate expression and can be regulated as such and the final

answer will be sensible even as the regulator is removed. It is easy to see that such a regulated $F(k)$ must be peaked near $k = 0$, and therefore short-ranged in k space. Thus we may perform a gradient expansion for $\Delta(\mathbf{R}, \mathbf{k})$ in k space, as in previous sections. Introducing $\mathbf{k} = (\mathbf{k}_1 + \mathbf{k}_2)/2$, $\mathbf{k}' = \mathbf{k}_2 - \mathbf{k}_1$, we obtain

$$\frac{\beta}{A^2} \int d\mathbf{R} \sum_{\mathbf{k}_1 \mathbf{k}_2} \Delta(\mathbf{R}, \mathbf{k}_1) \overline{\Delta}(\mathbf{R}, \mathbf{k}_2) F(|\mathbf{k}_1 - \mathbf{k}_2|)$$

$$= \frac{\beta}{A^2} \int d\mathbf{R} \sum_{\mathbf{k} \mathbf{k}'} (|\Delta(\mathbf{R}, \mathbf{k})|^2 - \frac{|\mathbf{k}'|^2}{4} |\nabla_{\mathbf{k}} \Delta(\mathbf{R}, \mathbf{k})|^2 + \cdots) F(k')$$

$$= \frac{\beta}{A} \sum_{\mathbf{k}} \int d\mathbf{R} [B_1 |\Delta(\mathbf{R}, \mathbf{k})|^2 + B_2 |\nabla_{\mathbf{k}} \Delta(\mathbf{R}, \mathbf{k})|^2 + \cdots], \tag{4.15}$$

where

$$B_1 = \frac{1}{A} \sum_{\mathbf{k}} F(k) = \frac{1}{V(0)}, \tag{4.16}$$

$$B_2 = -\frac{1}{A} \sum_{\mathbf{k}} F(k) \frac{k^2}{4} = \frac{1}{4} \nabla^2 \left(\frac{1}{V(r)} \right) |_{r=0}. \tag{4.17}$$

For $V(r) = V_0 e^{-r^2/2L^2}$, we have

$$B_1 = \frac{1}{V_0}, \tag{4.18}$$

$$B_2 = \frac{1}{2V_0 L^2}. \tag{4.19}$$

Clearly, the coefficient B_2 controls the fluctuations of Δ in the $\mathbf{k}$ space, which describes the internal degrees of freedom for the pairing amplitude; the larger the interaction range L is, the softer such fluctuations are. On the other hand, in the short range limit $L \to 0$, such fluctuations get completely suppressed, and the order parameter Δ depends only of the center of mass coordinate $\mathbf{R}$, and we recover the standard Ginzburg-Landau theory in terms of $\Delta(\mathbf{R})$ with no $\mathbf{k}$ dependence.[24] It is clear however, we need to keep full $\mathbf{k}$ dependence of Δ in the present case.

In the first term of Eq. (4.9), there is no mechanism to control the spatial ($\mathbf{R}$) fluctuations of Δ. Such fluctuations are controlled by the second term, which we now turn to. The second term in Eq. (4.9) takes the form

$$- \int d\mathbf{x}_1 d\mathbf{y}_1 d\mathbf{x}_2 d\mathbf{y}_2 Q(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2) \Delta(\mathbf{x}_1, \mathbf{y}_1) \overline{\Delta}(\mathbf{x}_2, \mathbf{y}_2)$$

$$= -\frac{1}{A} \sum_{\mathbf{k}} \int d\mathbf{R}_1 d\mathbf{R}_2 \Delta(\mathbf{R}_1, \mathbf{k}) \overline{\Delta}(\mathbf{R}_2, \mathbf{k}) P(\mathbf{R}_2 - \mathbf{R}_1, \mathbf{k}), \tag{4.20}$$

where

$$P(\mathbf{R}_2 - \mathbf{R}_1, \mathbf{k}) = \frac{1}{A} \sum_{\mathbf{k}_1, i\omega_n} G_0(\mathbf{k}_1, i\omega_n) G_0(2\mathbf{k} + \mathbf{k}_1, -i\omega_n) e^{i(2\mathbf{k} + 2\mathbf{k}_1) \cdot (\mathbf{R}_2 - \mathbf{R}_1)}, \tag{4.21}$$

and

$$G_0(\mathbf{k}, i\omega_n) = \frac{1}{i\omega_n - \epsilon_{\mathbf{k}}}. \tag{4.22}$$

In this term there is coupling between Δ's with different $\mathbf{R}$'s, but no coupling between Δ's with different $\mathbf{k}$'s. Thus the first and second quadratic terms in the action control the internal ($\mathbf{k}$) and spatial ($\mathbf{R}$) fluctuations of the order parameter Δ respectively.

The fluctuations of the overall magnitude of the order parameter are controlled by higher order terms in the effective action. As usual we may stop at the quartic term near T_c^{MF}, and neglect terms that involve the spatial gradient of the order parameter at this order.[24] For the present problem, we obtain (in a way similar to Ref. 24)

$$S_e^{(4)} = \frac{1}{8A} \sum_{\mathbf{k}} \int d\mathbf{R} |\Delta(\mathbf{R}, \mathbf{k})|^4 \left(\frac{\tanh(\beta \epsilon_{\mathbf{k}}/2)}{\epsilon_{\mathbf{k}}^3} - \frac{\beta}{2\epsilon_{\mathbf{k}}^2 \cosh^2(\beta \epsilon_{\mathbf{k}}/2)} \right). \tag{4.23}$$

Now we perform a spatial gradient expansion in $\mathbf{R}$, for the second quadratic term, Eq. (4.20). To do that, we first go to momentum space, and define

$$\Delta(\mathbf{Q},\mathbf{k}) = \int d\mathbf{R}\, e^{-i\mathbf{Q}\cdot\mathbf{R}}\Delta(\mathbf{R},\mathbf{k}). \tag{4.24}$$

In terms of $\Delta(\mathbf{Q},\mathbf{k})$, the second quadratic term reads

$$\frac{1}{A^2}\sum_{\mathbf{Q},\mathbf{k}} D(\mathbf{Q},\mathbf{k})|\Delta(\mathbf{Q},\mathbf{k})|^2, \tag{4.25}$$

where

$$\begin{aligned}
D(\mathbf{Q},\mathbf{k}) &= \sum_{i\omega_n} G_0(\mathbf{Q}/2-\mathbf{k}, i\omega_n)G_0(\mathbf{Q}/2+\mathbf{k}, -i\omega_n) \\
&= \frac{\beta}{2}\frac{\tanh(\beta\epsilon_{\mathbf{Q}/2-\mathbf{k}}/2) + \tanh(\beta\epsilon_{\mathbf{Q}/2+\mathbf{k}}/2)}{\epsilon_{\mathbf{Q}/2-\mathbf{k}} + \epsilon_{\mathbf{Q}/2+\mathbf{k}}}.
\end{aligned} \tag{4.26}$$

Expanding for small Q, we obtain

$$D(\mathbf{Q},\mathbf{k}) \approx D(0,\mathbf{k}) - \frac{\beta^3 v_F^2 \sinh(\beta\epsilon_{\mathbf{k}}/2)}{32\epsilon_{\mathbf{k}}\cosh^3(\beta\epsilon_{\mathbf{k}}/2)}(\mathbf{Q}\cdot\hat{\mathbf{k}})^2. \tag{4.27}$$

Fourier transforming back to real ($\mathbf{R}$) space, we obtain

$$-\frac{1}{A}\sum_{\mathbf{k}} D(0,\mathbf{k})\int d\mathbf{R}|\Delta(\mathbf{R},\mathbf{k})|^2 + \frac{1}{A}\sum_{\mathbf{k}}\frac{\beta^3 v_F^2 \sinh(\beta\epsilon_{\mathbf{k}}/2)}{32\epsilon_{\mathbf{k}}\cosh^3(\beta\epsilon_{\mathbf{k}}/2)}\int d\mathbf{R}|\hat{\mathbf{k}}\cdot\nabla\Delta(\mathbf{R},\mathbf{k})|^2. \tag{4.28}$$

Thus putting all these terms together, and keeping leading gradient terms only, we obtain

$$\begin{aligned}
S_e[\Delta,\overline{\Delta}] = \frac{\beta}{A}\sum_{\mathbf{k}}\int d\mathbf{R}\{&[B_1 - C(\beta,\mathbf{k})]|\Delta(\mathbf{R},\mathbf{k})|^2 + B_2|\nabla_{\mathbf{k}}\Delta(\mathbf{R},\mathbf{k})|^2 \\
&+ \frac{1}{2m_{\mathbf{k}}}|\hat{\mathbf{k}}\cdot\nabla\Delta(\mathbf{R},\mathbf{k})|^2 + U(k)|\Delta(\mathbf{R},\mathbf{k})|^4\},
\end{aligned} \tag{4.29}$$

where

$$B_1 = 1/V_0, \tag{4.30}$$
$$B_2 = 1/(2V_0 L^2), \tag{4.31}$$
$$C(\beta,k) = \tanh\frac{\beta\epsilon_k}{2}/(2\epsilon_k), \tag{4.32}$$
$$\frac{1}{2m_{\mathbf{k}}} = \frac{\beta^2 v_F^2 \sinh\frac{\beta\epsilon_k}{2}}{32\epsilon_k\cosh^3\frac{\beta\epsilon_k}{2}}, \tag{4.33}$$
$$U(k) = \frac{\tanh\frac{\beta\epsilon_k}{2}}{8\epsilon_k^3} - \frac{\beta}{16\epsilon_k^2\cosh^2\frac{\beta\epsilon_k}{2}}. \tag{4.34}$$

Eq. (4.29) is one of the central results of this paper; it is the basis for the calculation of reduction of T_c from T_c^{MF} due to thermal fluctuations of collective modes in the next section.

Thus far in our discussion we have assumed a uniform space with no background electromagnetic field. In the presence of a weak and slowly varying background magnetic field, using standard arguments[24] one finds that the only modification one needs to make in (4.29) is to replace ∇ by $\nabla + 2ie\mathbf{A}(\mathbf{R})/c$. The supercurrent is

$$\mathbf{j}(\mathbf{R}) = -c\frac{\delta S_e}{\delta\mathbf{A}(\mathbf{R})} = \frac{ie\beta}{A}\sum_{\mathbf{k}}\frac{\mathbf{k}}{m_k}\{\overline{\Delta}[\mathbf{k}\cdot(\nabla + \frac{2ie\mathbf{A}}{c})\Delta] - c.c.\}. \tag{4.35}$$

¿From this we may obtain the equivalent of London's equation:

$$\mathbf{j}(\mathbf{R}) = \frac{-4\beta e^2}{Ac} \sum_{\mathbf{k}} |\Delta_{\mathbf{k}}|^2 \frac{\mathbf{k}}{m_k} [\mathbf{k} \cdot \mathbf{A}(\mathbf{R})] = \{\frac{-4e^2}{Acd} \sum_{\mathbf{k}} \frac{|\Delta_{\mathbf{k}}|^2 k^2}{m_k}\} \mathbf{A}(\mathbf{R}), \tag{4.36}$$

from which the expression for penetration depth follows:

$$\frac{1}{\lambda^2} = \frac{8\beta e^2}{Adc^2} \sum_{\mathbf{k}} \frac{|\Delta_{\mathbf{k}}|^2 k^2}{m_k}. \tag{4.37}$$

Here d is the spacing between layers. These are, of course, mean-field results; fluctuations have been left out at this level. And they apply only near T_c^{MF}, even at the mean-field level.

V. REDUCTION OF T_C DUE TO THERMAL FLUCTUATIONS OF COLLECTIVE MODES AND PSEUDOGAP BEHAVIOR

We study in this section the reduction of of T_c from T_c^{MF} due to the fluctuations of the collective modes, using the Ginzburg-Landau free energy derived earlier. As a warm-up, as well as for the purpose of later comparison, we study the same effect in a weak coupling BCS superconductor with short range (δ-function) interaction, whose Ginzburg-Landau free energy is the familiar $O(N)$ ϕ^4 theory with $N = 2$:[24]

$$S_e = \beta \int d\mathbf{r} \left(-\frac{1}{4m_e} |\nabla \psi|^2 + a|\psi|^2 + b|\psi|^4 \right), \tag{5.1}$$

where in 3D we have

$$\psi(\mathbf{r}) = \left(\frac{7\zeta(3) m_e k_F \epsilon_F}{12\pi^4 (T_c^{MF})^2} \right)^{1/2} \Delta(\mathbf{r}), \tag{5.2}$$

$$a = \frac{6\pi^2 T_c^{MF}(T - T_c^{MF})}{7\zeta(3)\epsilon_F}, \tag{5.3}$$

$$b = \frac{9\pi^4 (T_c^{MF})^2}{14\zeta(3) m_e k_F \epsilon_F^2}, \tag{5.4}$$

and m_e is the effective mass of the electron.

In the mean field theory, one neglects the quartic term in (5.4), and T_c is determined by the point where a turns negative, which is nothing but T_c^{MF}. The fluctuation effects of the quartic term may be studied using the self-consistent field approximation that is exact in the limit $N \to \infty$.[25] Within this approximation, one replaces $|\psi|^4$ by $4\langle|\psi|^2\rangle|\psi|^2$, and requires that the following self-consistency equation is satisfied:

$$\langle|\psi|^2\rangle = \frac{1}{V} \sum_{\mathbf{k}} \langle \psi_{\mathbf{k}} \psi_{-\mathbf{k}} \rangle = \frac{1}{(2\pi)^3} \int d^3\mathbf{k} \frac{k_B T}{\frac{k^2}{4m_e} + a + 4b\langle|\psi|^2\rangle}. \tag{5.5}$$

At $T = T_c$, we ought to have $a + 4b\langle|\psi|^2\rangle = 0$. Thus the equation that determines T_c becomes

$$a + 4b \frac{4m_e k_B T_c}{(2\pi)^3} \int_{k < \Lambda_{uv}} \frac{d^3\mathbf{k}}{k^2} = 0, \tag{5.6}$$

where $\Lambda_{uv} \sim \hbar\omega_D/v_F$ is an appropriate high momentum (ultraviolet) cutoff for k. From this we obtain

$$\frac{T_c^{MF} - T_c}{T_c^{MF}} \sim \frac{k_B T_c \hbar \omega_D}{\epsilon_F^2} \ll 1 \tag{5.7}$$

for weak coupling superconductors, whence it is clear that the mean field value for T_c is extremely accurate.

In the following we demonstrate that the situation is very different, and T_c becomes significantly lower than T_c^{MF} when $L \gg \xi$, due to thermal fluctuations of the low energy collective modes discussed in previous sections. We use the Ginzburg-Landau theory appropriate for this situation, Eq. (4.29), and use the same self-consistent field approximation as above by replacing $|\Delta(\mathbf{R}, \mathbf{k})|^4$ with $4\langle|\Delta(\mathbf{R}, \mathbf{k})|^2\rangle|\Delta(\mathbf{R}, \mathbf{k})|^2$, after which the reduced action takes a quadratic form:

13

$$S_R[\Delta, \overline{\Delta}] = \frac{\beta}{A} \sum_{\mathbf{k}} \int d\mathbf{R} \left\{ [B_1 - C(\beta, \mathbf{k}) + \tilde{U}(k)]|\Delta(\mathbf{R}, \mathbf{k})|^2 + B_2 |\nabla_{\mathbf{k}} \Delta(\mathbf{R}, \mathbf{k})|^2 + \frac{1}{2m_{\mathbf{k}}} |\hat{\mathbf{k}} \cdot \nabla \Delta(\mathbf{R}, \mathbf{k})|^2 \right\}, \qquad (5.8)$$

where

$$\tilde{U}(k) = 4U(k)\langle |\Delta(\mathbf{R}, \mathbf{k})|^2 \rangle. \qquad (5.9)$$

We would like to formally diagonalise this quadratic form. To do that, we take advantage of the translation invariance of the action, and express the reduced action in terms of the Fourier transform of $\Delta(\mathbf{R}, \mathbf{k})$, $\Delta(\mathbf{Q}, \mathbf{k})$:

$$S_R[\Delta, \overline{\Delta}] = \frac{\beta}{A^2} \sum_{\mathbf{k}, \mathbf{Q}} \left\{ \left[B_1 - C(\beta, \mathbf{k}) + \tilde{U}(k) + \frac{(\hat{\mathbf{k}} \cdot \mathbf{Q})^2}{2m_k} \right] |\Delta(\mathbf{Q}, \mathbf{k})|^2 + B_2 |\nabla_{\mathbf{k}} \Delta(\mathbf{Q}, \mathbf{k})|^2 \right\}. \qquad (5.10)$$

Note that modes with different $\mathbf{Q}$'s decouple. To proceed further, we introduce eigen modes $\psi_{mn}(\mathbf{Q}, \mathbf{k})$ which satisfy

$$\left[B_1 - C(\beta, \mathbf{k}) + \tilde{U}(k) + \frac{(\hat{\mathbf{k}} \cdot \mathbf{Q})^2}{2m_k} - B_2 \nabla_{\mathbf{k}}^2 \right] \psi_{mn}(\mathbf{Q}, \mathbf{k}) = E_{mn}(Q) \psi_{mn}(\mathbf{Q}, \mathbf{k}), \qquad (5.11)$$

and the normalization condition:

$$\frac{1}{A} \sum_{\mathbf{k}} |\psi_{mn}(\mathbf{Q}, \mathbf{k})|^2 = 1. \qquad (5.12)$$

Here m and n are quantum numbers to be specified later. Expanding $\Delta(\mathbf{Q}, \mathbf{k})$ in terms of $\psi_{mn}(\mathbf{Q}, \mathbf{k})$:

$$\Delta(\mathbf{Q}, \mathbf{k}) = \sum_{mn} a_{mn}(\mathbf{Q}) \psi_{mn}(\mathbf{Q}, \mathbf{k}), \qquad (5.13)$$

we bring the reduced action to diagonal form:

$$S_R[\Delta, \overline{\Delta}] = \frac{\beta}{A} \sum_{\mathbf{Q}mn} E_{mn}(Q) |a_{mn}(\mathbf{Q})|^2. \qquad (5.14)$$

The self-consistent equation now becomes

$$\tilde{U}(k) = 4U(k)\langle |\Delta(\mathbf{R}, \mathbf{k})|^2 \rangle = \frac{4U(k)}{A^2} \sum_{\mathbf{Q}mn} |\psi_{mn}(\mathbf{Q}, \mathbf{k})|^2 \langle |a_{mn}(\mathbf{Q})|^2 \rangle = \frac{4U(k)}{A\beta} \sum_{\mathbf{Q}mn} \frac{|\psi_{mn}(\mathbf{Q}, \mathbf{k})|^2}{E_{mn}(Q)}. \qquad (5.15)$$

At $T = T_c$, we have the lowest eigenvalue

$$E_{00}(Q = 0) = 0; \qquad (5.16)$$

This poses a self-consistent condition that determines T_c.

The Schroedinger-like equation (5.11) is quite difficult to solve in general, one of the reason being the isotropy in $\mathbf{k}$ space is lost due to the presence of the term proportional to $(\hat{\mathbf{k}} \cdot \mathbf{Q})^2$. In order to proceed, we need to make a number of further approximations and assumptions. Firstly, we average $(\hat{\mathbf{k}} \cdot \mathbf{Q})^2$ along different directions of $\mathbf{k}$ and replace it by $Q^2/2$, so that isotropy in $\mathbf{k}$ space is restored and all modes may be labeled by an "angular momentum" quantum number m. We also notice that $-C(\beta, k)$ has its minimum at $k = k_F$, and goes to zero rapidly for $\epsilon_k > 1/\beta$. Hence we may approximate it by a "square" well, with depth $-C(\beta, k_F) = \frac{\beta}{4}$ and width $\frac{2}{v_F \beta}$. $U(k)$ (and therefore $\tilde{U}(k)$) has a similar structure, and we can also approximate it by a step-like structure, with height $U_{k_F} = \frac{\beta^3}{96}$ and the same width $\frac{2}{v_F \beta}$. Anticipating that the "low-energy" (E) modes will be localized in the step well, we may approximate $\frac{1}{2m_k}$ by a constant $\frac{1}{2m_{k_F}}$. With these simplifications, Eq. (5.11) reduces to that of the Schroedinger equation of a particle confined to a step potential well. Imposing periodic boundary condition at the ends of the well,[26] we obtain

$$\psi_{mn}(\mathbf{Q}, \mathbf{k}) = \sqrt{\frac{\pi v_F \beta}{k_F}} e^{i[2\pi m\theta_{\mathbf{k}} + n\pi(k - k_F)/v_F \beta]}, \qquad (5.17)$$

14

and at $T = T_c$ (using (5.16))

$$E_{mn}(Q) = \frac{\beta^3 v_F^2 Q^2}{64} + B_2(\frac{m^2}{k_F^2} + \pi^2 n^2 v_F^2 \beta^2). \tag{5.18}$$

Combining Eqs. (5.15) and (5.18), we obtain the equation that determines T_c:

$$B_1 - \frac{\beta_c}{4} + \frac{2}{3\pi k_F v_F} \sum_{mn} \int \frac{d\mathbf{Q}}{Q^2 + \frac{64 B_2}{\beta_c v_F^2 k_F^2}(m^2 + \pi^2 n^2 \beta_c^2 v_F^2 k_F^2)} = 0. \tag{5.19}$$

We notice that for the case of $m = 0$ and $n = 0$, the integral is logarithmically divergent at both infrared and ultraviolet. As discussed earlier, there is always an ultraviolet cutoff $\Lambda_{uv} \approx E_c/v_F$. The infrared divergence is a signature of the fact that 2D is the lower dimensionality for ordering of this model (we do not go into details of the Kosterlitz-Thouless picture for a true phase transition for $N = 2$ here). An infrared cutoff Λ_{ir} is provided by invoking the quasi-2D nature of all real systems, which eventually crossover to 3D at sufficiently long length scales. Assuming Λ_{ir} is sufficiently large compared to the energies (measured in proper units) of the modes whose fluctuations contribute significantly to the reduction of T_c, we obtain

$$\beta_c \approx \frac{4M}{3 v_F k_F} \log \frac{\Lambda_{uv}}{\Lambda_{ir}}, \tag{5.20}$$

where M is the number of modes contributing in the sum of m and n.

To determine M, we note that the self-consistent potential (for $Q = 0$; finite Q only adds a constant to it) is zero inside the well, and B_1 outside it. Thus in order for the previous approximations for mode solutions and "energies" to be valid, we need to have $E_{mn}(Q = 0) < B_1$. Summing up the number of these modes, we obtain

$$M \sim k_F \sqrt{B1/B_2} = k_F L. \tag{5.21}$$

We thus find

$$T_c = k_B/\beta_c \sim \frac{v_F}{L \log \frac{\Lambda_{uv}}{\Lambda_{ir}}} \sim \frac{T_c^{MF}}{(L/\xi) \log \frac{\Lambda_{uv}}{\Lambda_{ir}}}. \tag{5.22}$$

It clearly goes to zero as $L \to \infty$.[27] We would like to emphasize that here we have made a number of crude approximations in the calculation of T_c, thus the dependence of T_c on the range L may not be quantitatively reliable. However it is quite clear that the approximations we made tend to *underestimate* the importance of the fluctuation effects; thus the qualitative conclusion that $T_c \to 0$ as $L \to \infty$ must hold.

VI. SUMMARY AND DISCUSSION

In the bulk of the paper, we have taken the attitude that the model we study here, namely a system with long (compared to the coherence length) but finite range pairing interaction, is a theoretical model that is interesting in its own right, and worked out some unusual properties of this model, with emphasis on those properties that are qualitatively different from those of the standard weak coupling BCS superconductors stabilized by a short-range pairing potential. Our most interesting finding is that in this model the transition temperature T_c is controlled by thermal fluctuations of collective modes, and can be significantly lower than the quasiparticle gap Δ or the mean-field transition temperature T_c^{MF}; as a consequence in the temperature range $T_c < T < T_c^{MF}$ the system exhibits pseudogap behavior as the electrons are still paired while there is no superconducting long-range order.[28] In this section we attempt to make contact between our results and the phenomenology of cuprate superconductors, discuss the relation between our model and existing theoretical work on the pseudogap behavior, point out the limitations of our model as well as of our analysis, and indicate some natural extensions and directions for future study.

One of the motivations of the present study is the observation that the coherence length ξ is much shorter in the cuprates than in conventional superconductors. We are, however, by no means the first to suggest that a short ξ can lead to behavior qualitatively different from weak coupling BCS theory. Since the early days of high T_c, following the work of Leggett and Nozieres and Schmitt-Rink,[29] Randeria and coworkers,[30–32,5] as well as others,[33,34] have argued that the short coherence length may bring the cuprates to a regime that is intermediate between the weak-coupling BCS limit and the Bose-Einstein Condensation (BEC) limit of Cooper pairs. The latter case is realized if ξ is much shorter than the inter-particle spacing so that $k_F \xi \gg 1$; in terms of energy scales, that corresponds to

the case $\Delta \gg E_F$. In this case the Cooper pairs are so closely bound that they hardly overlap, and may be viewed as point-like hard-core bosons at low energies, while the transition temperature T_c is essentially their Bose-Einstein condensation temperature which is much lower than and unrelated to the pairing energy Δ. For $T_c < T < \Delta$, the electrons remain paired yet there is no long-range superfluid order, hence pseudogap behavior. In terms of low-energy excitations responsible for destroying superconductivity, in the BEC limit it is the linear Goldstone mode (assuming no Coulomb interaction), while the quasiparticle excitations (broken pairs) cost too much energy to have any effect on T_c. Put differently, the thermal fluctuations of this Goldstone mode are the classical phase fluctuations of the superconducting order parameter that control the transition in this limit. (We should note that Emery and Kivelson[13] have argued that classical phase fluctuations *are* the physics of the pseudogap regime in the underdoped materials, but their point of departure is logically distinct, deriving from a small zero temperature superfluid stiffness that is connected to the physics of a doped Mott insulator.) In the cuprates, $k_F \xi \sim 10$, and Δ is still a small energy compared to E_F (although the difference is not nearly as overwhelming as in conventional superconductors); we are thus still somewhat distant from the BEC limit. The interesting new feature of the model studied here is that by having ξ much smaller than the range of pairing interaction L, we can get the pseudogap behavior while staying the *weak coupling* regime (in the sense $k_F \xi \gg 1$ and $\Delta/E_F \ll 1$). In this case the Goldstone mode is unable to drive T_c below the scale of Δ by itself; it needs all the help from the other collective modes supported by this model. The low-energy spectra of the weak-coupling BCS superconductor, the BEC superconductor, and the present model are summarized schematically in Fig. 1. While there are clearly qualitative differences between the BEC (as well as phase fluctuation) picture and the present model, they share the common spirit that T_c is determined by collective modes instead of quasiparticle excitations. It is also worth mentioning that in real systems with long-range Coulomb interaction, the energy of the Goldstone mode is expected to be pushed up to the plasmon frequency; the importance of this fact to the thermal fluctuations of this mode (or classical phase fluctuations) is still under discussion.[35] On the other hand the Coulomb interaction is not expected to significantly affect the spectra of the (gapped) exciton-like modes discussed here. This is due to the different nature of the Goldstone mode and exciton modes: the former is a consequence of the broken gauge symmetry, and introducing the Coulomb interaction (or, more generally, the electromagnetic interaction) converts the Goldstone mechanism of broken symmetry to the Higgs mechanism. The exciton modes, on the other hand, are the bound states formed by quasiparticle pairs due to the residual attractive interaction between quasiparticles not included in the mean field approximation. We therefore believe the presence of Coulomb interaction does not affect the fluctuation physics discussed here significantly.

As pointed out already in the Introduction, in some theoretical models for cuprate superconductivity, the range of the interaction that gives rise to Cooper pairing can be very long. In the interlayer pair hopping model,[2,14] it is assumed that pairing is induced by a pair hopping term in the Hamiltonian, that is diagonal in momentum space. Fourier-transforming to real space, this corresponds to an *infinite* range hopping term for Cooper pairs, corresponding (loosely) to the $L \to \infty$ limit of the model we study here (the fact that the hopping term is off-diagonal in layer index is of no qualitative consequence). In that *specific form*, it has already been shown[36] that the model may be solved exactly in the absence of any other in-plane paring interaction, the transition temperature is zero, and there is pseudogap behavior at low temperatures. Our results are in agreement with this observation. The model we use here, however, is more general and versatile, and in particular enables one to address how the pseudogap behavior develops as the range L increases, and how T_c approaches zero as $L \to \infty$.

It is equally interesting to scrutinize the results obtained here in the context of the spin fluctuation theory[4] of cuprate superconductivity. In this model the range of the interaction L is essentially the spin-spin correlation length.[37] Thus L is of order lattice spacing in the overdoped region of the phase diagram, increases as the doping level x decreases, becomes much longer than the lattice spacing in the underdoped region, and eventually diverges upon approaching the antiferromagnetic phase boundary at very low doping. The pseudogap behavior is observed in the underdoped regime, where $T_c \to 0$ while the gap (the maximum of the d-wave gap measured at very low T by, say, photoemission) slowly *increases* as x decreases. This behavior is certainly consistent with our findings here; the reduction of T_c and its departure from the gap is due to the increase of L, while the size of gap, which in our model saturates at the depth of the potential well V_0 for large L, would be set by the scale of the near neighbor spin coupling strength J. Hence at a very crude level, we can qualitatively account for the phenomenology of the underdoped cuprates by combining our results with the spin-fluctuation model.

On the experimental side, some circumstantial evidence for both the importance of the longer range part (beyond near-neighbor) of the pairing interaction, and the possible relevance of our results, exists. i) Recent photoemission measurements[38] of gap anisotropy have found deviations from the standard

$$\Delta_{\mathbf{k}} \propto \cos k_x - \cos k_y \qquad (6.1)$$

dependence of the $d_{x^2-y^2}$ order parameter in the underdoped region, with the deviation increasing with the decreasing doping level. This deviation is interpreted[38] as due to longer-ranged pairing interaction, as a nearest-neighbor

attraction leads to Eq. (6.1). We consider this to be (somewhat indirect) evidence that the range of the pairing interaction increases with decreasing doping level in underdoped cuprates. (ii) It has been recently noticed[39] that there is a strong correlation between T_c and peak width of the *normal state* spin susceptibility $\chi(q,\omega)$, in some cuprates. Specifically, Balatsky and and Bourges[39] found that in $YBCO_{123}$ and La_{214} compounds, $T_c \propto \delta q$, where δq is the width of $Im\chi(q,\omega)$ at low ω. This lead the authors to conclude that "antiferromagnetism is likely responsible for the high T_c superconducting mechanism". If so, one is lead to the conclusion that $T_c \propto 1/L$, where $L \sim 1/\delta q$ is the range of the pairing interaction mediated by the *normal state* antiferromagnetic spin fluctuations, in agreement with our Eq. (5.22)! However, given the facts that our model is greatly oversimplified as far as the cuprates go (see below) and our treatment of the fluctuation effects is still preliminary (also below), we consider this agreement to be fortuitous at this point. Nevertheless, the findings of Ref. 39 do indicate the importance of the range of interaction on T_c, and are in qualitative agreement with our results. (iii) Experimentally, it has been found[40] that upon cooling underdoped cuprate samples, the temperature T^* at which the pseudogap opens up depends on the location in momentum space; T^* is highest near the superconducting gap maxima, while lowest near the gap nodes; it thus suggests that T^* is a "local" property in momentum space. This can be understood quite naturally by invoking a long range pairing interaction as studied here, since such an interaction is localized in momentum space. As we have shown in section III, one can define a momentum-dependent mean field transition temperature, $T_c^{MF}(\epsilon_\mathbf{k})$, which increases monotonically with $\Delta_\mathbf{k}$; it also sets the temperature $T^* \sim T_c^{MF}(\epsilon_\mathbf{k})$ below which a local gap is opened up and the pseudogap sets in.

We must emphasize that the contact between our model and the cuprate physics made above is tentative, and in its present form this model can only be viewed as an interesting toy model that gives rise to pseudogap behavior. In the following we discuss some limitations of the model as well as our treatment, and some natural directions for extensions.

The ground state of our model is a fully-gapped *s*-wave superconductor, while the cuprates (at least most of them) are known to have a *d*-wave order parameter. The most important difference between the two is that the latter supports *gapless* nodal quasiparticles. It has already been suggested[41] that the thermally excited quasiparticles may be responsible for emptying the superfluid stiffness, setting the scale of T_c, and giving rise to pseudogap behavior in underdoped cuprates. This important piece of physics is missing in our model. On the other hand, as long as collective modes of the *order parameter* are concerned, our method can be generalized to *d*-wave (or other unconventional superconductors) fairly easily. The key feature of a long (but finite) range pairing interaction is its *locality* in momentum space, i.e., $V_\mathbf{k}$ is sharply peaked in $\mathbf{k}$ space. This allows a gradient expansion in momentum space for the order parameter. In an *s*-wave superconductor, $V_\mathbf{k}$ is sharply peaked at $\mathbf{k} = 0$, while for a *d*-wave superconductor appropriate for cuprates, one would choose a $V_\mathbf{k}$ that is sharply peaked at a wave vector $\mathbf{Q}$ which is at or near (π,π). This, however, is not going to make any qualitative difference to the analysis made in this paper. At a more detailed level, two cases need to be distinguished from each other. (i) There is Fermi surface nesting and $\mathbf{Q}$ is exactly or very close to the nesting wave vector. In this case the entire Fermi surface participates in pairing actively and the analysis carried out here, based on the gradient expansion of the superconducting order parameter both along and perpendicular to the Fermi surface, carries through straightforwardly. (ii) There is no Fermi surface nesting, or $\mathbf{Q}$ is not close to the nesting wave vector. In this case only certain "hot spots" at the Fermi surfaces that are connected by $\mathbf{Q}$ participate in the pairing actively. In this case one can still develop a gradient expansion within the hot spots. In either case the qualitative features of our results are expected to be robust. We note that in a recent study of the spin-fermion-hot spot model,[42] it has been conjectured that in the limit of infinite spin-spin correlation length, the transition temperature obtained from solving the Eliashberg equation is only the onset of pseudogap behavior, while the real T_c vanishes in that limit. This conjecture is in agreement with our results, as the spin-spin correlation length corresponds to the range of interaction L in our model. In the present paper, we have developed a systematic way to study the physics of pseudogap behavior due to large L, and shown *explicitly* that $T_c \to 0$ as $L \to \infty$.

Our analysis of the thermal fluctuations of the collective modes is based on a Ginzburg-Landau free energy functional, which we derive using a functional integration formalism and an expansion in power series of the magnitude of the order parameter. Strictly speaking this power series expansion is valid only in the vicinity of T_c^{MF}, where the amplitude of the order parameter just starts to develop; at $T = T_c \ll T_c^{MF}$, the amplitude of the order parameter is big and such an expansion is no longer appropriate. What is also missing are the contributions from the fluctuations of components of the order parameter with finite Matsubara frequencies; we neglected them on the ground that we are primary interested effects of thermal fluctuations; however at low T these components are important to the physics, especially if one is also interested in the quantum fluctuations of the order parameter. Even with these simplifications, the resultant free energy functional is quite complicated, and we need to introduce approximations in the one-loop calculation of T_c, like neglecting the anisotropy in the internal (relative) space of the order parameter when the center of mass carries a finite momentum. It is quite possible that one can do a better job in analyzing the fluctuation, especially in the limit of $L/\xi \to \infty$. It is worth noting, however, that all our approximations tend to *underestimate* the effects of fluctuations, and thus our basic conclusion that $T_c \ll T_c^{MF} \sim \Delta$ when $L/\xi \gg 1$, is clearly valid.

Despite the various disclaimers made above, we hope the model and the crossover introduced here will serve as a

different paradigm of a non-BCS transition to a superconducting state, which can exhibit pseudogap behavior even without leaving the weak coupling regime. Its relevance to cuprate physics is not completely clear at present, but there are encouraging signs that it is worth further pursuit.

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[15] *E.g.*, in exciton condensates where electrons and holes form Cooper pairs due to the long-range Coulomb attraction, although the strictly Coulomb case in d=3 has a bound state spectrum with a distinct lowest energy state.

[16] See section II for an appropriate definition of ξ for the Cooper problem.

[17] Since our study is motivated by the cuprate superconductors which are layered compounds, throughout the paper we mostly assume that we are dealing with a 2D system. However in our analysis of the fluctuations in Section V we will invoke the quasi two-dimensional nature of the system to regulate the "Mermin-Wagner" infrared singularities by introducing an infrared cutoff on the 2D momenta.

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[23] If the interactions between electrons with same spin are kept, one needs to introduce two more sets of Hubbard-Stratonovich fields (see later), one for up-spin pairs and one for down-spin pairs, to decouple these interactions. This will add some more technical complexity to the analysis, but does not change the essential physics.

[24] See e.g. V. N. Popov, *Functional Integrals and Collective Excitations*, Cambridge University Press, Cambridge (1987).

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[26] It is clear that a more appropriate boundary condition is to require the modes to vanish at the boundaries. However the

periodic boundary condition has the advantage that the mode amplitude is a constant inside the well, thus $\tilde{U}(k)$ is also a constant inside the well, consistent with our assumption.

[27] This is the only place the cutoff E_c explicitly enters a physical quantity (through Λ_{uv}), T_c, in this paper. However the dependence is so weak (logarithmic), that the dependence of E_c on L hardly affects anything.

[28] A longer range pairing interaction can give rise to other interesting physical effects not discussed in this paper. For example, it has been argued recently (H. Ghosh, Phys. Rev. B **60**, 6814 (1999)) that it can induce a mixture of order parameters of different symmetry at mean field level in certain models.

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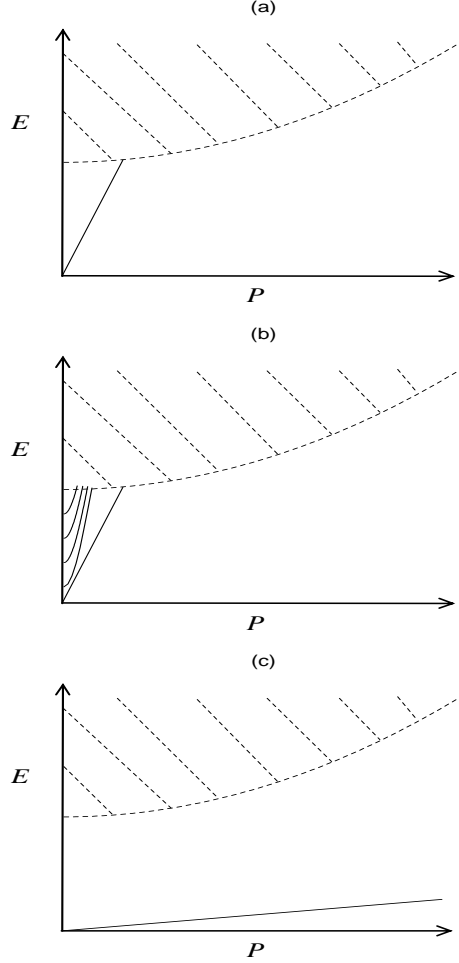[42] Ar. Abanov, A. V. Chubukov, and A. M. Finkel'stein, cond-mat/9911445.

FIG. 1. Schematic illustration of the neutral excitation spectra of three different types of superconductors, in the absence of Coulomb interactions. The region shaded by dashed lines with a gap 2Δ is the quasiparticle pair continuum, common to all three cases; the solid lines stand for collective modes. For a weak-coupling BCS superconductor with short-range pairing interaction (part (a); $\xi \gg 1/k_F$ and $\xi \gg L$, where ξ is the coherence length, k_F is the Fermi wave vector, and L is the range of the pairing interaction), the only collective excitation is the linear Goldstone mode, whose velocity ($v \sim v_F$) is very big as compared to the scale of the gap: $vk_F \gg 2\Delta$, thus contributes little to the thermodynamics and hence the determination of T_c. In the BEC case (part (c); $\xi \ll 1/k_F$), again the linear Goldstone mode is the only low-energy collective mode, but due to the reversed energy scale: $vk_F \ll 2\Delta$, it dominates the thermodynamics and sets the scale for T_c. The situation studied in this paper, as illustrated in part (b), is in some sense between these two extremes, namely we have $\xi \gg 1/k_F$ as in weak-coupling BCS, but in the meantime $\xi \ll L$. Here the Goldstone mode is very steep and hence contributes little to the thermodynamics; however the the combined effect of all the low-energy modes, stabilized by the condition $\xi \ll L$, dominates the thermodynamics and determines T_c.